

# Generalized Sampling Expansions Associated with Quaternion Fourier Transform

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## Abstract

Quaternion-valued signals along with quaternion Fourier transforms (QFT) provide an effective framework for vector-valued signal and image processing. However, the sampling theory of quaternion valued signals has not been well developed. In this paper, we present the generalized sampling expansions associated with QFT by using the generalized translation and convolution. We show that a  $\sigma$ -bandlimited quaternion valued signal in QFT sense can be reconstructed from the samples of output signals of  $M$  linear systems based on QFT. Quaternion linear canonical transform (QLCT) is a generalization of QFT with six parameters. Using the relationship between QFT, we derive the sampling formula for  $\sigma$ -bandlimited quaternion-valued signal in QLCT sense. Examples are given to illustrate our results.

**Keywords:** Quaternion Fourier transform; quaternion linear canonical transform; sampling expansions; generalized translation; convolution theorem

## 1 Introduction

Generalized sampling expansions (GSE) developed by Papoulis [1] indicates that a  $\sigma$ -bandlimited signal can be reconstructed from the samples of output signals of  $M$  linear systems. Namely,

$$f(t) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^M g_m(nT) y_m(t - nT)$$

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where  $g_m$  ( $1 \leq m \leq M$ ) are output signals of  $M$  linear systems,  $y_m$  ( $1 \leq m \leq M$ ) are determined by a linear simultaneous equations whose coefficients are generated by system functions. Some classical sampling expansions, for instance, Shannon sampling expansions are special cases of Papoulis' result by choosing specific systems.

Over the years, the GSE has been extended in different ways. Hoskins and Pinto [2] extended the GSE to bandlimited distribution functions. A multidimensional extension of GSE was introduced by Cheung [3] for real-valued functions. While, Wei, Ran and Li [4] presented the GSE with generalized integral transformation, such as fractional Fourier transform. In this paper, higher-dimensional extension of GSE to quaternion-valued functions are studied. By powerful modelling of rotation and orientation, quaternion have shown advantages in physical and engineering applications such as computer graphics [5, 6] and robotics [7]. Furthermore, QFT has been regarded as a useful analysis tool in color image and signal processing [8, 9, 10, 11, 12] in recently years. Therefore, it is desirable to define a system based on QFT to analyze quaternion-valued signals. Moreover, it is worthwhile and interesting to investigate the GSE using the samples of output signals of  $M$  linear systems based on QFT. However, for the non-commutativity of the quaternion multiplication, the desirable shift property of classical Fourier transform is no longer available for QFT. Meanwhile, an crucial tool in signal processing called convolution theorem does not hold for QFT as well. The purpose of this paper is to overcome these problems and investigate the GSE. In this paper, we propose a novel translation of quaternion-valued signals and apply it to deduce the convolution theorem of QFT. More importantly, we present the GSE associated with QFT by proposed translation and convolution.

The rest of the paper is organized as follows. In the next section, we review QFT and some of its properties such as Plancherel theorem. Section 3 proposes a new translation and its corresponding convolution theorem. In Section 4, we present the generalized sampling expansion of bandlimited quaternion-valued signals in the sense of QFT. In Section 5, examples are presented to illustrate our results. In Section 6, we further discuss the sampling formula for  $\sigma$ -bandlimited quaternion valued signal in QLCT sense.

## 2 Preliminary

### 2.1 Quaternion algebra

Let's recall quaternion algebra  $\mathbb{H} := \{q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 : q_0, q_1, q_2, q_3 \in \mathbb{R}\}$ , where the imaginary elements  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  obey  $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ . For every quaternion  $q = q_0 + \underline{q}$ ,  $\underline{q} = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$ , the scalar and vector parts of  $q$ , are defined as  $\text{Sc}(q) = q_0$  and  $\text{Vec}(q) = \underline{q}$ , respectively. If  $q = \text{Vec}(q)$ , then  $q$  is called pure imaginary quaternion. The quaternion conjugate is defined by  $\bar{q} = q_0 - \underline{q} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3$ , and the norm  $|q|$  of  $q$  defined as  $|q|^2 = q\bar{q} = \bar{q}q = \sum_{m=0}^{m=3} q_m^2$ . Then we have

$$\bar{\bar{q}} = q, \quad \overline{p + q} = \bar{p} + \bar{q}, \quad \overline{pq} = \bar{q} \bar{p}, \quad |pq| = |p||q|, \quad \forall p, q \in \mathbb{H}.$$

Using the conjugate and norm of  $q$ , one can define the inverse of  $q \in \mathbb{H} \setminus \{0\}$  as  $q^{-1} = \bar{q}/|q|^2$ .

The quaternion exponential function  $\mathbf{e}^q$  is defined by means of an infinite series as  $\mathbf{e}^q := \sum_{n=0}^{\infty} \frac{q^n}{n!}$ . Analogous to the complex case one may derive a closed-form representation:  $\mathbf{e}^q = \mathbf{e}^{q_0}(\cos |q| + \frac{q}{|q|} \sin |q|)$ .

Let  $\mathbf{X}$  be a Lebesgue measurable subset of  $\mathbb{R}^2$ , the left  $\mathbb{H}$ -module  $L^p(\mathbf{X}, \mathbb{H})$  ( $p = 1, 2$ ) consists of all  $\mathbb{H}$ -valued functions whose  $p$ th power is Lebesgue integrable on  $\mathbf{X}$ . The left quaternionic inner product of  $f, g \in L^2(\mathbf{X}, \mathbb{H})$  is defined by

$$\langle f, g \rangle := \int_{\mathbf{X}} f(x_1, x_2) \overline{g(x_1, x_2)} dx_1 dx_2.$$

In fact,  $L^2(\mathbf{X}, \mathbb{H})$  is a left quaternionic Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  (see [13]). Therefore, if  $\{e_n\}$  is an orthonormal basis of  $L^2(\mathbf{X}, \mathbb{H})$ , then

$$\langle f, g \rangle = \sum_n \langle f, e_n \rangle \langle e_n, g \rangle \quad (2.1)$$

holds for all  $f, g \in L^2(\mathbf{X}, \mathbb{H})$ . This is a desirable property of quaternionic Hilbert space [14].

## 2.2 Quaternion Fourier transform

The quaternion Fourier transform (QFT) was first introduced by Ell to analyze partial differential equations [15]. Since then, QFT were applied to color image processing effectively [8, 9, 10, 12]. There are different types of QFT [16] due to the non-commutativity of the quaternion multiplication. In [17], the authors investigated the properties of distinct types of QFT thoroughly, especially the following right-sided QFT.

**Definition 2.1 (QFT)** For every  $f \in L^1(\mathbb{R}^2, \mathbb{H})$ , the right-sided QFT of  $f$  is defined by

$$(\mathcal{F}f)(\omega_1, \omega_2) := \frac{1}{2\pi} \int_{\mathbb{R}^2} f(x_1, x_2) \mathbf{e}^{-i\omega_1 x_1} \mathbf{e}^{-j\omega_2 x_2} dx_1 dx_2.$$

If  $\mathcal{F}f$  is also in  $L^1(\mathbb{R}^2, \mathbb{H})$ , the inversion QFT formula (see [18, 17]) holds, that is

$$f(x_1, x_2) = (\mathcal{F}^{-1}\mathcal{F}f)(x_1, x_2) := \frac{1}{2\pi} \int_{\mathbb{R}^2} (\mathcal{F}f)(\omega_1, \omega_2) \mathbf{e}^{j\omega_2 x_2} \mathbf{e}^{i\omega_1 x_1} d\omega_1 d\omega_2,$$

for almost every  $(x_1, x_2) \in \mathbb{R}^2$ . By Plancherel theorem (see [18, 17]), the QFT can be extended to  $L^2(\mathbb{R}^2, \mathbb{H})$ . As an operator on  $L^2(\mathbb{R}^2, \mathbb{H})$ , the QFT  $\Psi$  is a bijection and the Parseval's identity  $\|\Psi f\|_2 = \|f\|_2$  holds.

**Remark 2.2** Since  $\Psi(\Psi^{-1})$  coincides with  $\mathcal{F}(\mathcal{F}^{-1})$  in  $L^1(\mathbb{R}^2, \mathbb{H}) \cap L^2(\mathbb{R}^2, \mathbb{H})$ . For simplicity of notations, in the following, by capital letter  $F$ , we mean the QFT of  $f \in L^1(\mathbb{R}^2, \mathbb{H}) \cup L^2(\mathbb{R}^2, \mathbb{H})$  if no otherwise specified.

### 3 Generalized translation and convolution

A generalized translation related to the general integral transform with kernel  $K(\omega, t)$  was introduced in [19]. Motivated by this study, we define the generalized translation to the quaternion-valued signals.

**Definition 3.1** Let  $f \in L^1(\mathbb{R}^2, \mathbb{H}) \cup L^2(\mathbb{R}^2, \mathbb{H})$  and  $F \in L^1(\mathbb{R}^2, \mathbb{H})$ . The generalized translation related to QFT is defined by

$$f(x_1 \ominus y_1, x_2 \ominus y_2) := \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathbf{e}^{-\mathbf{i}\omega_1 y_1} \mathbf{e}^{-\mathbf{j}\omega_2 y_2} F(\omega_1, \omega_2) \mathbf{e}^{\mathbf{j}\omega_2 x_2} \mathbf{e}^{\mathbf{i}\omega_1 x_1} d\omega_1 d\omega_2. \quad (3.1)$$

**Remark 3.2** In the complex case, the 2D Fourier transform of  $f(x_1 - y_1, x_2 - y_2)$  with respect to  $(x_1, x_2)$  is  $\mathbf{e}^{-\mathbf{i}\omega_1 y_1} \mathbf{e}^{-\mathbf{i}\omega_2 y_2} \hat{f}(\omega_1, \omega_2)$ , where  $\hat{f}(\omega_1, \omega_2)$  is the 2D Fourier transform of complex-valued function  $f$ . Therefore, the generalized translation  $f(x_1 \ominus y_1, x_2 \ominus y_2)$ , in some sense, is an analogue of  $f(x_1 - y_1, x_2 - y_2)$ . Moreover,  $f(x_1 \ominus y_1, x_2 \ominus y_2)$  coincides with  $f(x_1 - y_1, x_2 - y_2)$  for some special  $f(x_1, x_2)$  (see Example 5.1).

Suppose that  $h \in L^1(\mathbb{R}^2, \mathbb{H}) \cup L^2(\mathbb{R}^2, \mathbb{H})$ ,  $H \in L^1(\mathbb{R}^2, \mathbb{H})$  and  $f \in L^1(\mathbb{R}^2, \mathbb{H})$ , then

$$\int_{\mathbb{R}^4} |f(y_1, y_2) H(\omega_1, \omega_2)| dy_1 dy_2 d\omega_1 d\omega_2 < \infty.$$

Therefore, by Fubini's Theorem, we have

$$\begin{aligned} & \frac{1}{2\pi} \int_{\mathbb{R}^2} f(y_1, y_2) (x_1 \ominus y_1, x_2 \ominus y_2) dy_1 dy_2 \\ &= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} dy_1 dy_2 f(y_1, y_2) \int_{\mathbb{R}^2} \mathbf{e}^{-\mathbf{i}\omega_1 y_1} \mathbf{e}^{-\mathbf{j}\omega_2 y_2} H(\omega_1, \omega_2) \mathbf{e}^{\mathbf{j}\omega_2 x_2} \mathbf{e}^{\mathbf{i}\omega_1 x_1} d\omega_1 d\omega_2 \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} F(\omega_1, \omega_2) H(\omega_1, \omega_2) \mathbf{e}^{\mathbf{j}\omega_2 x_2} \mathbf{e}^{\mathbf{i}\omega_1 x_1} d\omega_1 d\omega_2 \\ &= (\mathcal{F}^{-1}G)(x_1, x_2), \end{aligned} \quad (3.2)$$

where  $G = FH$ . Thus it is reasonable to define the following generalized convolution.

**Definition 3.3** Let  $f, h, G = FH \in L^1(\mathbb{R}^2, \mathbb{H}) \cup L^2(\mathbb{R}^2, \mathbb{H})$ . The convolution of  $f$  and  $h$  is defined by

$$(f \star h)(x_1, x_2) := (\mathcal{F}^{-1}G)(x_1, x_2).$$

Obviously, The QFT of  $f \star h$  is  $FH$ .

**Theorem 3.4** If any of the following conditions is satisfied.

1. If  $h \in L^1(\mathbb{R}^2, \mathbb{H}) \cup L^2(\mathbb{R}^2, \mathbb{H})$ ,  $H \in L^1(\mathbb{R}^2, \mathbb{H})$  and  $f \in L^1(\mathbb{R}^2, \mathbb{H})$ .
2. If  $h \in L^1(\mathbb{R}^2, \mathbb{H}) \cup L^2(\mathbb{R}^2, \mathbb{H})$ ,  $H \in L^1(\mathbb{R}^2, \mathbb{H}) \cap L^2(\mathbb{R}^2, \mathbb{H})$  and  $f \in L^2(\mathbb{R}^2, \mathbb{H})$ .

Then

$$(f \star h)(x_1, x_2) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(y_1, y_2)(x_1 \ominus y_1, x_2 \ominus y_2) dy_1 dy_2.$$

**Proof.** The first case is obviously true. We now prove the second case. Since  $f \in L^2(\mathbb{R}^2, \mathbb{H})$ , then  $F \in L^2(\mathbb{R}^2, \mathbb{H})$  by Plancherel theorem. Moreover, there is a sequence  $\{f_n\}$  in  $L^1(\mathbb{R}^2, \mathbb{H}) \cap L^2(\mathbb{R}^2, \mathbb{H})$  converging to  $f$  in  $L^2$  norm such that  $F = \lim_{n \rightarrow \infty} F_n = \lim_{n \rightarrow \infty} \mathcal{F}f_n$ . Note that  $H \in L^2(\mathbb{R}^2, \mathbb{H})$ . Therefore, by Hölder inequality,  $FH \in L^1(\mathbb{R}^2, \mathbb{H})$  and

$$\begin{aligned} & (f \star h)(x_1, x_2) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} F(\omega_1, \omega_2) H(\omega_1, \omega_2) \mathbf{e}^{j\omega_2 x_2} \mathbf{e}^{i\omega_1 x_1} d\omega_1 d\omega_2 \\ &= \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{\mathbb{R}^2} F_n(\omega_1, \omega_2) H(\omega_1, \omega_2) \mathbf{e}^{j\omega_2 x_2} \mathbf{e}^{i\omega_1 x_1} d\omega_1 d\omega_2 \\ &= \lim_{n \rightarrow \infty} \frac{1}{4\pi^2} \int_{\mathbb{R}^4} dy_1 dy_2 f_n(y_1, y_2) \mathbf{e}^{-i\omega_1 y_1} \mathbf{e}^{-j\omega_2 y_2} H(\omega_1, \omega_2) \mathbf{e}^{j\omega_2 x_2} \mathbf{e}^{i\omega_1 x_1} d\omega_1 d\omega_2 \\ &= \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{\mathbb{R}^2} f_n(y_1, y_2)(x_1 \ominus y_1, x_2 \ominus y_2) dy_1 dy_2 \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} f(y_1, y_2)(x_1 \ominus y_1, x_2 \ominus y_2) dy_1 dy_2. \end{aligned} \tag{3.3}$$

The interchange of integral and limit is permissible for the continuity of inner product.  $\square$

## 4 The GSE associated with QFT

We give a definition of bandlimited signals in QFT sense.

**Definition 4.1 (bandlimited)** A signal  $f(x_1, x_2)$  is  $\sigma$ -bandlimited in QFT sense if it can be expressed as

$$f(x_1, x_2) = \frac{1}{2\pi} \int_I F(\omega_1, \omega_2) \mathbf{e}^{j\omega_2 x_2} \mathbf{e}^{i\omega_1 x_1} d\omega_1 d\omega_2$$

where  $F \in L^2(I, \mathbb{H})$  and  $I = [-\sigma, \sigma]^2$ . For any  $\sigma > 0$ , denote by  $\mathbf{B}_\sigma^q$  the totality of the  $\sigma$ -bandlimited signals in QFT sense.

If  $f \in \mathbf{B}_\sigma^q$ , by Plancherel theorem, we have  $f \in L^2(\mathbb{R}^2, \mathbb{H})$  and the QFT of  $f$  is  $F$ . In this part, we show that  $f$  can be reconstructed from the samples of the inverse QFT of  $M := m^2$  functions  $G_k = FH_k$ , ( $k = 1, 2, \dots, M$ ) if  $H_k$  satisfy suitable conditions.

Let  $T := \frac{m\pi}{\sigma}$ ,  $c := \frac{2\sigma}{m} = \frac{2\pi}{T}$  and

$$I_{n_1 n_2} := [-\sigma + (n_1 - 1)c, -\sigma + n_1 c] \times [-\sigma + (n_2 - 1)c, -\sigma + n_2 c]. \tag{4.1}$$

Then

$$f(x_1, x_2) = \sum_{n_1=0}^{m-1} \sum_{n_2=0}^{m-1} \frac{1}{2\pi} \int_{I_{11}} F(\omega_1 + n_1 c, \omega_2 + n_2 c) \mathbf{e}^{\mathbf{j}(\omega_2 + n_2 c)x_2} \mathbf{e}^{\mathbf{i}(\omega_1 + n_1 c)x_1} d\omega_1 d\omega_2. \quad (4.2)$$

To state our results, we need some further notations. Let

$$\begin{cases} a_{n_1 n_2}(\omega_1, \omega_2) := F(\omega_1 + (n_1 - 1)c, \omega_2 + (n_2 - 1)c), \\ b_{n_1 n_2}(\omega_1, \omega_2, x_1, x_2) := \mathbf{e}^{\mathbf{j}(\omega_2 + (n_2 - 1)c)x_2} \mathbf{e}^{\mathbf{i}(\omega_1 + (n_1 - 1)c)x_1}, \\ r_{n_1 n_2}^k(\omega_1, \omega_2) := H_k(\omega_1 + (n_1 - 1)c, \omega_2 + (n_2 - 1)c). \end{cases}$$

They form the following vectors or matrices:

$$\begin{cases} \vec{F}(\omega_1, \omega_2) := (A(1, :), A(2, :), \dots, A(m, :))^T, \\ \vec{E}(\omega_1, \omega_2, x_1, x_2) := (B(1, :), B(2, :), \dots, B(m, :))^T, \\ \vec{H}_k(\omega_1, \omega_2) := (R_k(1, :), R_k(2, :), \dots, R_k(m, :))^T, \\ \underline{H}(\omega_1, \omega_2) := (\vec{H}_1(\omega_1, \omega_2), \vec{H}_2(\omega_1, \omega_2), \dots, \vec{H}_M(\omega_1, \omega_2)), \\ \vec{G}(\omega_1, \omega_2) := (\tilde{G}_1(\omega_1, \omega_2), \tilde{G}_2(\omega_1, \omega_2), \dots, \tilde{G}_M(\omega_1, \omega_2)) = \vec{F}(\omega_1, \omega_2)^T \underline{H}(\omega_1, \omega_2), \end{cases}$$

where  $A$ ,  $B$ ,  $R_k$  are  $m \times m$  matrices with entries  $a_{n_1 n_2}(\omega_1, \omega_2)$ ,  $b_{n_1 n_2}(\omega_1, \omega_2, x_1, x_2)$ ,  $r_{n_1 n_2}^k(\omega_1, \omega_2)$  respectively. Assume that  $\underline{H}$  is invertible for every  $(\omega_1, \omega_2) \in I_{11}$ . Denote the inverse of  $\underline{H}$  by

$$\underline{H}^{-1}(\omega_1, \omega_2) := (\vec{Q}_1(\omega_1, \omega_2); \vec{Q}_2(\omega_1, \omega_2); \dots; \vec{Q}_M(\omega_1, \omega_2))$$

where  $\vec{Q}_k(\omega_1, \omega_2) = (Q_k(1, :), Q_k(2, :), \dots, Q_k(m, :))$  and  $Q_k = (q_{n_1 n_2}^k(\omega_1, \omega_2))_{m \times m}$ . Then (4.2) becomes

$$f(x_1, x_2) = \frac{1}{2\pi} \int_{I_{11}} \vec{F}(\omega_1, \omega_2) \vec{E}(\omega_1, \omega_2, x_1, x_2) d\omega_1 d\omega_2.$$

In the complex case,  $\vec{E}$  only depends on  $(x_1, x_2)$  (see [3]). However, due to the non-commutativity of the quaternion algebra,  $\vec{E}$  depends on both  $(x_1, x_2)$  and  $(\omega_1, \omega_2)$  in quaternionic case.

**Lemma 4.2** *If  $H_k \in L^2(I, \mathbb{H})$ , then samples of the inverse QFT of  $G_k$  ( $k = 1, 2, \dots, M$ ) can be expressed as:*

$$g_k(n_1 T, n_2 T) = \frac{1}{2\pi} \int_{I_{11}} \tilde{G}_k(\omega_1, \omega_2) \mathbf{e}^{\mathbf{j}\omega_2 n_2 T} \mathbf{e}^{\mathbf{i}\omega_1 n_1 T} d\omega_1 d\omega_2,$$

where  $I_{11}$  is given in (4.1).

**Proof.** Since  $H_k \in L^2(I, \mathbb{H})$ , then  $G_k \in L^2(I, \mathbb{H})$ ,

$$\begin{aligned}
g_k(x_1, x_2) &= \frac{1}{2\pi} \int_I G_k(\omega_1, \omega_2) \mathbf{e}^{\mathbf{j}\omega_2 x_2} \mathbf{e}^{\mathbf{j}\omega_1 x_1} d\omega_1 d\omega_2 \\
&= \frac{1}{2\pi} \int_I F(\omega_1, \omega_2) H_k(\omega_1, \omega_2) \mathbf{e}^{\mathbf{j}\omega_2 x_2} \mathbf{e}^{\mathbf{j}\omega_1 x_1} d\omega_1 d\omega_2 \\
&= \sum_{l_1=1}^m \sum_{l_2=1}^m \frac{1}{2\pi} \int_{I_{11}} a_{l_1 l_2}(\omega_1, \omega_2) r_{l_1 l_2}^k(\omega_1, \omega_2) b_{l_1 l_2}(\omega_1, \omega_2, x_1, x_2) d\omega_1 d\omega_2 \\
&= \frac{1}{2\pi} \int_{I_{11}} \sum_{l_1=1}^m \sum_{l_2=1}^m a_{l_1 l_2}(\omega_1, \omega_2) r_{l_1 l_2}^k(\omega_1, \omega_2) b_{l_1 l_2}(\omega_1, \omega_2, x_1, x_2) d\omega_1 d\omega_2
\end{aligned}$$

and  $b_{l_1 l_2}(\omega_1, \omega_2, n_1 T, n_2 T) = \mathbf{e}^{\mathbf{j}\omega_2 n_2 T} \mathbf{e}^{\mathbf{j}\omega_1 n_1 T}$  which is independent of  $l_1 l_2$ . Therefore

$$\begin{aligned}
&\sum_{l_1=1}^m \sum_{l_2=1}^m a_{l_1 l_2}(\omega_1, \omega_2) r_{l_1 l_2}^k(\omega_1, \omega_2) b_{l_1 l_2}(\omega_1, \omega_2, x_1, x_2) \\
&= \vec{F}(\omega_1, \omega_2)^T \underline{\vec{H}}_k(\omega_1, \omega_2) \mathbf{e}^{\mathbf{j}\omega_2 n_2 T} \mathbf{e}^{\mathbf{j}\omega_1 n_1 T}.
\end{aligned}$$

It follows that

$$\begin{aligned}
g_k(n_1 T, n_2 T) &= \frac{1}{2\pi} \int_{I_{11}} \vec{F}(\omega_1, \omega_2)^T \underline{\vec{H}}_k(\omega_1, \omega_2) \mathbf{e}^{\mathbf{j}\omega_2 n_2 T} \mathbf{e}^{\mathbf{j}\omega_1 n_1 T} d\omega_1, \omega_2 \\
&= \frac{1}{2\pi} \int_{I_{11}} \tilde{G}_k(\omega_1, \omega_2) \mathbf{e}^{\mathbf{j}\omega_2 n_2 T} \mathbf{e}^{\mathbf{j}\omega_1 n_1 T} d\omega_1 \omega_2.
\end{aligned}$$

□

**Lemma 4.3** Suppose that  $q_{l_1 l_2}^k \in L^2(I_{11}, \mathbb{H})$  and let

$$\tilde{q}_{n_1 n_2}^k(\omega_1, \omega_2) = q_{n_1 n_2}^k(\omega_1 - (n_1 - 1)c, \omega_2 - (n_2 - 1)c) \chi_{I_{n_1 n_2}}(\omega_1, \omega_2)$$

and

$$Y_k(\omega_1, \omega_2) = \frac{T^2}{2\pi} \sum_{n_1=1}^m \sum_{n_2=1}^m \tilde{q}_{n_1 n_2}^k(\omega_1, \omega_2). \quad (4.3)$$

Then for every  $(n_1, n_2) \in \mathbb{Z}^2$ ,  $\frac{4\pi^2}{T^2} y_k(x_1 \ominus n_1 T, x_2 \ominus n_2 T)$  equals to

$$\int_{I_{11}} \mathbf{e}^{-\mathbf{j}\omega_1 n_1 T} \mathbf{e}^{-\mathbf{j}\omega_2 n_2 T} \sum_{l_1=1}^m \sum_{l_2=1}^m q_{l_1 l_2}^k(\omega_1, \omega_2) b_{l_1 l_2}(\omega_1, \omega_2, x_1, x_2) d\omega_1 \omega_2$$

where  $y_k$  is the inverse QFT of  $Y_k$ .

**Proof.** By rewriting  $y_k(x_1 \ominus n_1 T, x_2 \ominus n_2 T)$  in the form of (3.1) and substituting  $\mathbf{e}^{\mathbf{j}\omega_2 x_2} \mathbf{e}^{\mathbf{j}\omega_1 x_1}$  and  $\mathbf{e}^{-\mathbf{j}\omega_1 n_1 T} \mathbf{e}^{-\mathbf{j}\omega_2 n_2 T}$  with  $b_{l_1 l_2}(\omega_1 - (l_1 - 1)c, \omega_2 - (l_2 - 1)c, x_1, x_2)$  and

$$\overline{b_{n_1 n_2}(\omega_1 - (l_1 - 1)c, \omega_2 - (l_2 - 1)c, n_1 T, n_2 T)}$$

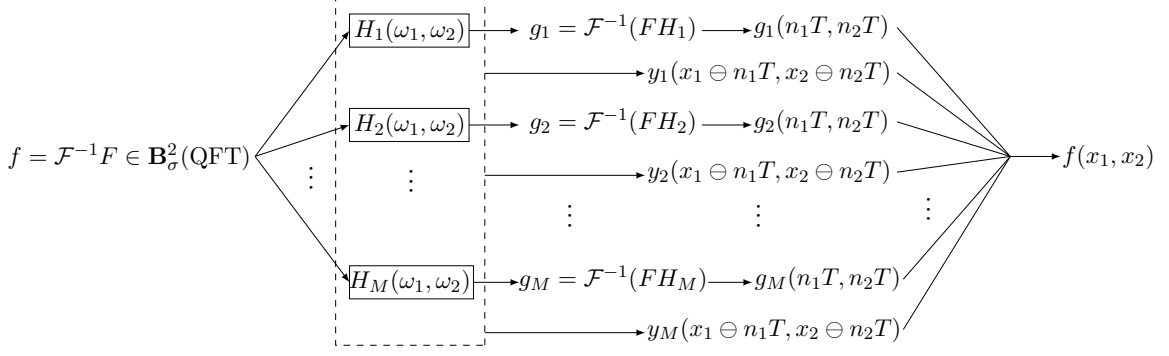


Figure 1: Diagram of GSE associated with QFT.

respectively, we obtain

$$\begin{aligned}
& \frac{2\pi}{T^2} \int_{\mathbb{R}^2} \mathbf{e}^{-\mathbf{i}\omega_1 n_1 T} \mathbf{e}^{-\mathbf{j}\omega_2 n_2 T} Y_k(\omega_1, \omega_2) \mathbf{e}^{\mathbf{j}\omega_2 x_2} \mathbf{e}^{\mathbf{i}\omega_1 x_1} d\omega_1 d\omega_2 \\
&= \sum_{l_1=1}^m \sum_{l_2=1}^m \int_{\mathbb{R}^2} \mathbf{e}^{-\mathbf{i}\omega_1 n_1 T} \mathbf{e}^{-\mathbf{j}\omega_2 n_2 T} \tilde{q}_{l_1 l_2}^k(\omega_1, \omega_2) \mathbf{e}^{\mathbf{j}\omega_2 x_2} \mathbf{e}^{\mathbf{i}\omega_1 x_1} d\omega_1 d\omega_2 \\
&= \sum_{l_1=1}^m \sum_{l_2=1}^m \int_{I_{l_1 l_2}} \overline{b_{n_1 n_2}(\omega_1 - (l_1 - 1)c, \omega_2 - (l_2 - 1)c, n_1 T, n_2 T)} \\
&\quad q_{l_1 l_2}^k(\omega_1 - (l_1 - 1)c, \omega_2 - (l_2 - 1)c) b_{l_1 l_2}(\omega_1 - (l_1 - 1)c, \omega_2 - (l_2 - 1)c, x_1, x_2) d\omega_1 d\omega_2 \\
&= \sum_{l_1=1}^m \sum_{l_2=1}^m \int_{I_{11}} \overline{b_{l_1 l_2}(\omega_1, \omega_2, n_1 T, n_2 T)} q_{l_1 l_2}^k(\omega_1, \omega_2) b_{l_1 l_2}(\omega_1, \omega_2, x_1, x_2) d\omega_1 d\omega_2 \\
&= \int_{I_{11}} \mathbf{e}^{-\mathbf{i}\omega_1 n_1 T} \mathbf{e}^{-\mathbf{j}\omega_2 n_2 T} \sum_{l_1=1}^m \sum_{l_2=1}^m q_{l_1 l_2}^k(\omega_1, \omega_2) b_{l_1 l_2}(\omega_1, \omega_2, x_1, x_2) d\omega_1 d\omega_2. \tag{4.4}
\end{aligned}$$

which completes the proof.  $\square$

Now we give the generalized sampling expansion associated with QFT.

**Theorem 4.4** *Let  $H_1, H_2, \dots, H_M$  such that*

1.  $H_k \in L^2(I, \mathbb{H})$ ,
2.  $\underline{H}(\omega_1, \omega_2)$  is invertible for every  $(\omega_1, \omega_2) \in I_{11}$  and  $q_{l_1 l_2}^k \in L^2(I_{11}, \mathbb{H})$ .

*Then  $f$  can be reconstructed from samples  $g_k(n_1 T, n_2 T)$  of*

$$\begin{aligned}
g_k(x_1, x_2) &= (f \star h_k)(x_1, x_2) \\
&= \frac{1}{2\pi} \int_I F(\omega_1, \omega_2) H_k(\omega_1, \omega_2) \mathbf{e}^{\mathbf{j}\omega_2 x_2} \mathbf{e}^{\mathbf{i}\omega_1 x_1} d\omega_1 d\omega_2.
\end{aligned}$$



More specifically,

$$f(x_1, x_2) = \sum_{k=1}^M \sum_{n_1, n_2} g_k(n_1 T, n_2 T) y_k(x_1 \ominus n_1 T, x_2 \ominus n_2 T) \quad (4.5)$$

where  $y_k$  is the inverse QFT of  $Y_k$ .

**Proof.** Since  $\underline{H}(\omega_1, \omega_2)$  is invertible for every  $(\omega_1, \omega_2) \in I_{11}$  then

$$\underline{H}^{-1}(\omega_1, \omega_2) \vec{E}(\omega_1, \omega_2, x_1, x_2)$$

is a  $m \times 1$  matrix and the  $k$ th element equals to

$$\sum_{l_1=1}^m \sum_{l_2=1}^m q_{l_1 l_2}^k(\omega_1, \omega_2) b_{l_1 l_2}(\omega_1, \omega_2, x_1, x_2).$$

Therefore

$$\begin{aligned} f(x_1, x_2) &= \frac{1}{2\pi} \int_{I_{11}} \vec{F}(\omega_1, \omega_2) \vec{E}(\omega_1, \omega_2, x_1, x_2) d\omega_1 d\omega_2 \\ &= \frac{1}{2\pi} \int_{I_{11}} \vec{G}(\omega_1, \omega_2) \underline{H}(\omega_1, \omega_2)^{-1} \vec{E}(\omega_1, \omega_2, x_1, x_2) d\omega_1 d\omega_2 \\ &= \sum_{k=1}^M \frac{1}{2\pi} \int_{I_{11}} \tilde{G}_k(\omega_1, \omega_2) \sum_{l_1=1}^m \sum_{l_2=1}^m q_{l_1 l_2}^k(\omega_1, \omega_2) b_{l_1 l_2}(\omega_1, \omega_2, x_1, x_2) d\omega_1 d\omega_2. \end{aligned}$$

As  $F, H_k \in L^2(I, \mathbb{H})$ , it is easy to see that  $\tilde{G}_k \in L^2(I_{11}, \mathbb{H})$ . Also, if  $q_{l_1 l_2}^k \in L^2(I, \mathbb{H})$  then

$$\sum_{l_1=1}^m \sum_{l_2=1}^m q_{l_1 l_2}^k(\omega_1, \omega_2) b_{l_1 l_2}(\omega_1, \omega_2, x_1, x_2) \in L^2(I_{11}, \mathbb{H})$$

for every  $(x_1, x_2) \in \mathbb{R}^2$ . Since  $\{\frac{T}{2\pi} \mathbf{e}^{-i\omega_1 n_1 T} \mathbf{e}^{-j\omega_2 n_2 T}\}_{(n_1, n_2) \in \mathbb{Z}^2}$  is an orthonormal basis of  $L^2(I_{11}, \mathbb{H})$ . Therefore by invoking (2.1) we have

$$\begin{aligned} &\frac{1}{2\pi} \int_{I_{11}} \tilde{G}_k(\omega_1, \omega_2) \sum_{l_1=1}^m \sum_{l_2=1}^m q_{l_1 l_2}^k(\omega_1, \omega_2) b_{l_1 l_2}(\omega_1, \omega_2, x_1, x_2) d\omega_1 d\omega_2 \\ &= \sum_{n_1, n_2} \frac{T^2}{8\pi^3} \int_{I_{11}} \tilde{G}_k(\omega_1, \omega_2) \mathbf{e}^{j\omega_2 n_2 T} \mathbf{e}^{i\omega_1 n_1 T} d\omega_1 d\omega_2 \\ &\quad \int_{I_{11}} \mathbf{e}^{-i\omega_1 n_1 T} \mathbf{e}^{-j\omega_2 n_2 T} \sum_{l_1=1}^m \sum_{l_2=1}^m q_{l_1 l_2}^k(\omega_1, \omega_2) b_{l_1 l_2}(\omega_1, \omega_2, x_1, x_2) d\omega_1 d\omega_2 \end{aligned}$$

Hence, by Lemma 4.2 and Lemma 4.3, we obtain (4.5).  $\square$

## 5 Examples

**Example 5.1** If  $f$  is  $\sigma$ -bandlimited, we have

$$f(x_1, x_2) = \sum_{n_1, n_2} f(n_1 T, n_2 T) \frac{\sin(\sigma x_1 - n_1 \pi) \sin(\sigma x_2 - n_2 \pi)}{(\sigma x_1 - n_1 \pi)(\sigma x_2 - n_2 \pi)} \quad (5.1)$$

by choosing  $m = 1$  and  $H_1(\omega_1, \omega_2) = 1$ , where  $T = \frac{\pi}{\sigma}$ . Let  $\sigma' = \rho\sigma$  with  $\rho > 1$  and  $H(\omega_1, \omega_2) = H^1(\omega_1)H^1(\omega_2)$  with

$$H^1(\omega_1) = \begin{cases} 1, & |\omega_1| \leq \sigma, \\ 0, & |\omega_1| \geq \sigma', \\ \frac{1}{(1-\rho)\sigma}|\omega_1| + \frac{\rho}{\rho-1}, & \sigma \leq |\omega_1| \leq \sigma'. \end{cases}$$

Note that  $f$  is  $\sigma'$ -bandlimited. Therefore, by applying Theorem 4.4 with  $M = 1$  and  $H(\omega_1, \omega_2)$  defined above, we have

$$f(x_1, x_2) = \sum_{n_1, n_2} f(n_1 T', n_2 T') y(x_1 - n_1 T, x_2 - n_2 T) \quad (5.2)$$

where

$$y(x_1, x_2) = \frac{4(\sin^2 \frac{\rho\sigma x_1}{2} - \sin^2 \frac{\sigma x_1}{2})(\sin^2 \frac{\rho\sigma x_2}{2} - \sin^2 \frac{\sigma x_2}{2})}{x_1^2 x_2^2 \rho^2 (\rho - 1)^2 \sigma^4}$$

and  $T' = \frac{\pi}{\sigma'} = \frac{T}{\rho} < T$ . For any fixed  $(x_1, x_2) \in \mathbb{R}^2$ , (5.2) converges faster than (5.1). It illustrates that convergence rate of sampling series can be enhanced by increasing the sampling frequency.

**Example 5.2** Express a  $\sigma$ -bandlimited function  $f(x_1, x_2)$  from the samples  $g(n_1 T, n_2 T)$  of the integral

$$g(x_1, x_2) = \frac{1}{2\pi} \int_I F(\omega_1, \omega_2) H(\omega_1, \omega_2) \mathbf{e}^{\mathbf{j}\omega_2 x_2} \mathbf{e}^{\mathbf{i}\omega_1 x_1} d\omega_1, \omega_2$$

where  $H(\omega_1, \omega_2) = \alpha\beta(\beta + \mathbf{j}\omega_2)^{-1}(\alpha + \mathbf{i}\omega_1)^{-1}(\alpha, \beta > 0)$ ,  $I = [-\sigma, \sigma]^2$ ,  $T = \frac{\pi}{\sigma}$ . In fact, by Theorem 3.4, we have

$$g(x_1, x_2) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(y_1, y_2) \tilde{h}(x_1 \ominus y_2, x_2 \ominus y_2) dy_1 dy_2$$

where

$$\tilde{h}(x_1 \ominus y_1, x_2 \ominus y_2) = \frac{1}{2\pi} \int_I \mathbf{e}^{-\mathbf{i}\omega_1 y_1} \mathbf{e}^{-\mathbf{j}\omega_2 y_2} H(\omega_1, \omega_2) \mathbf{e}^{\mathbf{j}\omega_2 x_2} \mathbf{e}^{\mathbf{i}\omega_1 x_1} d\omega_1 d\omega_2.$$

It is easy to see that  $H$  satisfies all conditions of Theorem 4.4. From (4.3), we have

$$Y(\omega_1, \omega_2) = \frac{T^2(\alpha + \mathbf{i}\omega_1)(\beta + \mathbf{j}\omega_2)}{2\pi\alpha\beta} \chi_I(\omega_1, \omega_2).$$

By direct computation,  $4\pi T^{-2}y(x_1 \ominus n_1 T, x_2 \ominus n_2 T)$  is given by

$$\begin{aligned}
& \frac{4 \sin \sigma(x_1 - n_1 T) \sin \sigma(x_2 - n_2 T)}{(x_1 - n_1 T)(x_2 - n_2 T)} \\
& - \frac{4 \sin \sigma(x_2 + n_2 T) [\sin \sigma(x_1 - n_1 T) - \sigma(x_1 - n_1 T) \cos \sigma(x_1 - n_1 T)]}{\alpha(x_2 + n_2 T)(x_1 - n_1 T)^2} \\
& - \frac{4 \sin \sigma(x_1 - n_1 T) [\sin \sigma(x_2 - n_2 T) - \sigma(x_2 - n_2 T) \cos \sigma(x_2 - n_2 T)]}{\beta(x_1 - n_1 T)(x_2 - n_2 T)^2} \\
& + \frac{4 [\sin \sigma(x_1 - n_1 T) - \sigma(x_1 - n_1 T) \cos \sigma(x_1 - n_1 T)]}{\alpha \beta (x_1 - n_1 T)^2 (x_2 + n_2 T)^2} \\
& [\sin \sigma(x_2 + n_2 T) - \sigma(x_2 + n_2 T) \cos \sigma(x_2 + n_2 T)].
\end{aligned}$$

**Example 5.3** Theorem 4.4 permits us to express a  $\sigma$ -bandlimited function  $f(x_1, x_2)$  from its samples and samples of its partial derivatives. Let  $m = 2$ ,  $T = \frac{2\pi}{\sigma}$ ,  $c = \sigma$  and  $H_1(\omega_1, \omega_2) = 1$ ,  $H_2(\omega_1, \omega_2) = \mathbf{i}\omega_1$ ,  $H_3(\omega_1, \omega_2) = \mathbf{j}\omega_2$ ,  $H_4(\omega_1, \omega_2) = \mathbf{k}\omega_1\omega_2$ . By [7], we have  $g_1(x_1, x_2) = f(x_1, x_2)$ ,  $g_2(x_1, x_2) = \frac{\partial f}{\partial x_1}(x_1, -x_2)$ ,  $g_3(x_1, x_2) = \frac{\partial f}{\partial x_2}(x_1, x_2)$ ,  $g_4(x_1, x_2) = -\frac{\partial^2 f}{\partial x_1 x_2}(x_1, -x_2)$ . Furthermore,

$$\underline{H} = \begin{pmatrix} 1 & \mathbf{i}\omega_1 & \mathbf{j}\omega_2 & \mathbf{k}\omega_1\omega_2 \\ 1 & \mathbf{i}\omega_1 & \mathbf{j}(\omega_2 + c) & \mathbf{k}\omega_1(\omega_2 + c) \\ 1 & \mathbf{i}(\omega_1 + c) & \mathbf{j}\omega_2 & \mathbf{k}(\omega_1 + c)\omega_2 \\ 1 & \mathbf{i}(\omega_1 + c) & \mathbf{j}(\omega_2 + c) & \mathbf{k}(\omega_1 + c)(\omega_2 + c) \end{pmatrix} = A_1 + A_2 \mathbf{j}$$

where  $A_1 = \begin{pmatrix} 1 & \mathbf{i}\omega_1 & 0 & 0 \\ 1 & \mathbf{i}\omega_1 & 0 & 0 \\ 1 & \mathbf{i}(\omega_1 + c) & 0 & 0 \\ 1 & \mathbf{i}(\omega_1 + c) & 0 & 0 \end{pmatrix}$  and  $A_2 = \begin{pmatrix} 0 & 0 & \omega_2 & \mathbf{i}\omega_1\omega_2 \\ 0 & 0 & (\omega_2 + c) & \mathbf{i}\omega_1(\omega_2 + c) \\ 0 & 0 & \omega_2 & \mathbf{i}(\omega_1 + c)\omega_2 \\ 0 & 0 & (\omega_2 + c) & \mathbf{i}(\omega_1 + c)(\omega_2 + c) \end{pmatrix}$ . The complex adjoint matrix [20] of  $\underline{H}$  denoted by  $C(\underline{H})$  is defined as

$$C(\underline{H}) = \begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix}.$$

Zhang [20] showed that  $\underline{H}$  is invertible if and only if  $C(\underline{H})$  is invertible. Moreover,  $C(\underline{H}^{-1}) = [C(\underline{H})]^{-1}$  if  $\underline{H}^{-1}$  exists. Since  $|C(\underline{H})| = c^8 \neq 0$  for every  $(\omega_1, \omega_2) \in I$ . Therefore  $\underline{H}$  is invertible for every  $(\omega_1, \omega_2) \in I$ . In fact,  $[C(\underline{H})]^{-1} = (U_1, U_2)$  where

$$U_1 = \begin{pmatrix} \frac{(c+\omega_2)(c+\omega_1)}{c^2} & -\frac{\omega_2(c+\omega_1)}{c^2} & -\frac{(c+\omega_2)\omega_1}{c^2} & \frac{\omega_1\omega_2}{c^2} \\ \frac{\mathbf{i}(c+\omega_2)}{c^2} & -\frac{\mathbf{i}\omega_2}{c^2} & -\frac{\mathbf{i}(c+\omega_2)}{c^2} & \frac{\mathbf{i}\omega_2}{c^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{c+\omega_1}{c^2} & \frac{c+\omega_1}{c^2} & \frac{\omega_1}{c^2} & -\frac{\omega_1}{c^2} \\ -\frac{\mathbf{i}}{c^2} & \frac{\mathbf{i}}{c^2} & \frac{\mathbf{i}}{c^2} & -\frac{\mathbf{i}}{c^2} \end{pmatrix}$$

and

$$U_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{c+\omega_1}{c^2} & -\frac{c+\omega_1}{c^2} & -\frac{\omega_1}{c^2} & \frac{\omega_1}{c^2} \\ -\frac{\mathbf{i}}{c^2} & \frac{\mathbf{i}}{c^2} & \frac{\mathbf{i}}{c^2} & -\frac{\mathbf{i}}{c^2} \\ \frac{(c+\omega_2)(c+\omega_1)}{c^2} & -\frac{\omega_2(c+\omega_1)}{c^2} & -\frac{(c+\omega_2)\omega_1}{c^2} & \frac{\omega_2\omega_1}{c^2} \\ -\frac{\mathbf{i}(c+\omega_2)}{c^2} & \frac{\mathbf{i}\omega_2}{c^2} & \frac{\mathbf{i}(c+\omega_2)}{c^2} & -\frac{\mathbf{i}\omega_2}{c^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus

$$\underline{H}^{-1} = \begin{pmatrix} \frac{(c+\omega_2)(c+\omega_1)}{c^2} & -\frac{\omega_2(c+\omega_1)}{c^2} & -\frac{(c+\omega_2)\omega_1}{c^2} & \frac{\omega_2\omega_1}{c^2} \\ \frac{\mathbf{i}(c+\omega_2)}{c^2} & -\frac{\mathbf{i}\omega_2}{c^2} & -\frac{\mathbf{i}(c+\omega_2)}{c^2} & \frac{\mathbf{i}\omega_2}{c^2} \\ \frac{\mathbf{j}(c+\omega_1)}{c^2} & -\frac{\mathbf{j}(c+\omega_1)}{c^2} & -\frac{\mathbf{j}\omega_1}{c^2} & \frac{\mathbf{j}\omega_1}{c^2} \\ -\frac{\mathbf{k}}{c^2} & \frac{\mathbf{k}}{c^2} & \frac{\mathbf{k}}{c^2} & -\frac{\mathbf{k}}{c^2} \end{pmatrix}$$

and hence

$$\begin{aligned} & \frac{1}{2\pi} Y_1(\omega_1, \omega_2) \\ &= \sigma^{-4}(c + \omega_2)(c + \omega_1) \chi_{I_{11}}(\omega_1, \omega_2) - \sigma^{-4}(\omega_2 - c)(c + \omega_1) \chi_{I_{12}}(\omega_1, \omega_2) \\ & \quad - \sigma^{-4}(c + \omega_2)(\omega_1 - c) \chi_{I_{21}}(\omega_1, \omega_2) + \sigma^{-4}(\omega_2 - c)(\omega_1 - c) \chi_{I_{22}}(\omega_1, \omega_2). \end{aligned}$$

By direct computation,

$$y_1(x_1 \ominus n_1 T, x_2 \ominus n_2 T) = \frac{16 \sin^2(\frac{\sigma}{2} x_1 - n_1 \pi) \sin^2(\frac{\sigma}{2} x_2 - n_2 \pi)}{\sigma^4 (x_1 - n_1 T)^2 (x_2 - n_2 T)^2}.$$

Similarly, we have

$$\begin{aligned} y_2(x_1 \ominus n_1 T, x_2 \ominus n_2 T) &= \frac{16 \sin^2(\frac{\sigma}{2} x_1 - n_1 \pi) \sin^2(\frac{\sigma}{2} x_2 + n_2 \pi)}{\sigma^4 (x_1 - n_1 T)(x_2 + n_2 T)^2}, \\ y_3(x_1 \ominus n_1 T, x_2 \ominus n_2 T) &= \frac{16 \sin^2(\frac{\sigma}{2} x_1 - n_1 \pi) \sin^2(\frac{\sigma}{2} x_2 - n_2 \pi)}{\sigma^4 (x_1 - n_1 T)^2 (x_2 - n_2 T)}, \\ y_4(x_1 \ominus n_1 T, x_2 \ominus n_2 T) &= -\frac{16 \sin^2(\frac{\sigma}{2} x_1 - n_1 \pi) \sin^2(\frac{\sigma}{2} x_2 + n_2 \pi)}{\sigma^4 (x_1 - n_1 T)(x_2 + n_2 T)}. \end{aligned}$$

## 6 Sampling theorem for Quaternion linear canonical transform

The right-sided quaternion linear canonical transform (QLCT) which is generalization of linear canonical transform (LCT) to quaternion algebra, was firstly studied in [21]. In this section, we investigate the sampling theory associated with QLCT. The right-sided QLCT of a signal  $f \in L^1(\mathbb{R}^2, \mathbb{H})$  with real matrix parameter  $A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathbb{R}^{2 \times 2}$  such

that  $\det(A_i) = 1$  for  $i = 1, 2$  is defined by [21]

$$(\mathcal{L}f)(\omega_1, \omega_2) := \int_{\mathbb{R}^2} f(x_1, x_2) K_{A_1}^{\mathbf{i}}(x_1, \omega_1) K_{A_2}^{\mathbf{j}}(x_2, \omega_2) dx_1 dx_2$$

where

$$K_{A_1}^{\mathbf{i}}(x_1, \omega_1) := \frac{1}{\sqrt{\mathbf{i}2\pi b_1}} \mathbf{e}^{\mathbf{i}(\frac{a_1}{2b_1}x_1^2 - \frac{1}{b_1}x_1\omega_1 + \frac{d_1}{2b_1}\omega_1^2)}, \quad \text{for } b_1 \neq 0 \quad (6.1)$$

and

$$K_{A_2}^{\mathbf{j}}(x_2, \omega_2) := \frac{1}{\sqrt{\mathbf{j}2\pi b_2}} \mathbf{e}^{\mathbf{j}(\frac{a_2}{2b_2}x_2^2 - \frac{1}{b_2}x_2\omega_2 + \frac{d_2}{2b_2}\omega_2^2)}, \quad \text{for } b_2 \neq 0. \quad (6.2)$$

Here,  $\frac{1}{\sqrt{\mu 2\pi b}}$  represents  $|2\pi b|^{\frac{-1}{2}} \mathbf{e}^{\mu \frac{\text{sgn} b - 2}{4} \pi}$  for any pure imaginary unit quaternion  $\mu$  and nonzero real number  $b$ .

If  $\mathcal{L}f$  is also in  $L^1(\mathbb{R}^2, \mathbb{H})$ , the the inversion QLCT formula [17] holds, that is

$$f(x_1, x_2) = (\mathcal{L}^{-1}\mathcal{L}f)(x_1, x_2) := \int_{\mathbb{R}^2} (\mathcal{L}f)(\omega_1, \omega_2) K_{A_2^{-1}}^{\mathbf{j}}(\omega_2, x_2) K_{A_1^{-1}}^{\mathbf{i}}(\omega_1, x_1) d\omega_1 d\omega_2,$$

for almost every  $(x_1, x_2) \in \mathbb{R}^2$ . By Plancherel theorem [17], the QLCT can be extended to  $L^2(\mathbb{R}^2, \mathbb{H})$ . As an operator on  $L^2(\mathbb{R}^2, \mathbb{H})$ , the QLCT  $\Phi$  is a bijection and the Parseval's identity  $\|\Phi f\|_2 = \|f\|_2$  holds. Since  $\Phi(\Phi^{-1})$  coincides with  $\mathcal{L}(\mathcal{L}^{-1})$  in  $L^1(\mathbb{R}^2, \mathbb{H}) \cap L^2(\mathbb{R}^2, \mathbb{H})$ . So we use letter  $F_{\mathbf{A}}$  to denote the QLCT of  $f \in L^1(\mathbb{R}^2, \mathbb{H}) \cup L^2(\mathbb{R}^2, \mathbb{H})$ .

In the classical case, the LCT is just a variation of the standard Fourier transform, some of its properties can be deduced from those of the Fourier transform by a change of variable. Moreover, the proofs of many sampling formulae associated with the LCT are somewhat based on those of the Fourier transform. In the quaternionic case, however,  $\mathcal{L}$  can not directly establish relation with  $\mathcal{F}$  as mentioned in [17]. So it is hard to derive GSE associated with the QLCT from existing results of the QFT. On the other hand, Lemma 4.2 is based on periodicity of kernel  $\mathbf{e}^{\mathbf{j}\omega_2 x_2} \mathbf{e}^{\mathbf{i}\omega_1 x_1}$ , but due to the quadratic term of kernel (6.1) and (6.2) in the QLCT, periodicity of kernel no longer possess. When  $M > 1$ , (4.5) of Theorem 4.4 is also called *multichannel sampling expansion*. In the following, we give a single-channel sampling expansion associated with the QLCT, that is we only focus on the case of  $M = 1$ .

Firstly, we introduce the generalized translation related to the QLCT:

$$f(x_1 \boxminus y_1, x_2 \boxminus y_2) \quad (6.3)$$

$$:= \int_{\mathbb{R}^2} \overline{K_{A_2^{-1}}^{\mathbf{j}}(\omega_2, y_2) K_{A_1^{-1}}^{\mathbf{i}}(\omega_1, y_1)} F_{\mathbf{A}}(\omega_2, \omega_2) K_{A_2^{-1}}^{\mathbf{j}}(\omega_2, x_2) K_{A_1^{-1}}^{\mathbf{i}}(\omega_1, x_1) d\omega_1 d\omega_2 \quad (6.4)$$

provided taht the right-hand side integral is well defined. Then we have the following theorem.

**Theorem 6.1** *Suppose that  $f$  is  $\sigma$ -bandlimited in QLCT sense, that is*

$$f(x_1, x_2) = \int_I F_{\mathbf{A}}(\omega_2, \omega_2) K_{A_2^{-1}}^{\mathbf{j}}(\omega_2, x_2) K_{A_1^{-1}}^{\mathbf{i}}(\omega_1, x_1) d\omega_1 d\omega_2$$

where  $F_{\mathbf{A}} \in L^2(I, \mathbb{H})$  and  $I = [-\sigma, \sigma]^2$ . Let

$$g(x_1, x_2) = \int_I F_{\mathbf{A}}(\omega_2, \omega_2) H(\omega_1, \omega_2) K_{A_2^{-1}}^{\mathbf{j}}(\omega_2, x_2) K_{A_1^{-1}}^{\mathbf{i}}(\omega_1, x_1) d\omega_1 d\omega_2$$

where  $H(\omega_1, \omega_2), H(\omega_1, \omega_2)^{-1} \in L^2(I, \mathbb{H})$ . Let

$$Y_{\mathbf{A}}(\omega_1, \omega_2) = T^2 |b_1 b_2| H(\omega_1, \omega_2)^{-1} \chi_I(\omega_1, \omega_2).$$

Then  $f$  can be reconstructed from samples  $g(n_1 b_1 T, n_2 b_2 T)$ :

$$f(x_1, x_2) = \sum_{n_1, n_2} g(n_1 b_1 T, n_2 b_2 T) y(x_1 \boxminus n_1 b_1 T, x_2 \boxminus n_2 b_2 T) \quad (6.6)$$

where  $y$  is the inverse QLCT of  $Y_{\mathbf{A}}$  and  $T = \frac{\pi}{\sigma}$ .

**Proof.** We note that  $\varphi_{n_1 n_2}(\omega_1, \omega_2) := T |b_1 b_2|^{\frac{1}{2}} \overline{K_{A_2^{-1}}^{\mathbf{j}}(\omega_2, n_2 b_2 T) K_{A_1^{-1}}^{\mathbf{i}}(\omega_1, n_1 b_1 T)}$  is an orthonormal basis of  $L^2(I, \mathbb{H})$ . Therefore by invoking (2.1) we have

$$\begin{aligned} f(x_1, x_2) &= \int_I F_{\mathbf{A}}(\omega_2, \omega_2) K_{A_2^{-1}}^{\mathbf{j}}(\omega_2, x_2) K_{A_1^{-1}}^{\mathbf{i}}(\omega_1, x_1) d\omega_1 d\omega_2 \\ &= \int_I F_{\mathbf{A}}(\omega_2, \omega_2) H(\omega_1, \omega_2) H(\omega_1, \omega_2)^{-1} K_{A_2^{-1}}^{\mathbf{j}}(\omega_2, x_2) K_{A_1^{-1}}^{\mathbf{i}}(\omega_1, x_1) d\omega_1 d\omega_2 \\ &= \sum_{n_1, n_2} \left( \int_I F_{\mathbf{A}}(\omega_2, \omega_2) H(\omega_1, \omega_2) \overline{\varphi_{n_1 n_2}(\omega_1, \omega_2)} d\omega_1 d\omega_2 \right) \\ &\quad \left( \int_I \varphi_{n_1 n_2}(\omega_1, \omega_2) H(\omega_1, \omega_2)^{-1} K_{A_2^{-1}}^{\mathbf{j}}(\omega_2, x_2) K_{A_1^{-1}}^{\mathbf{i}}(\omega_1, x_1) d\omega_1 d\omega_2 \right) \\ &= \sum_{n_1, n_2} g(n_1 b_1 T, n_2 b_2 T) y(x_1 \boxminus n_1 b_1 T, x_2 \boxminus n_2 b_2 T) \end{aligned}$$

which completes the proof.  $\square$

Now we give an example for this Theorem. Suppose that  $f$  is  $\sigma$ -bandlimited in QLCT sense and let  $H(\omega_1, \omega_2) = 1$ , then  $g(x_1, x_2) = f(x_1, x_2)$ . Therefore

$$f(x_1, x_2) = \sum_{n_1, n_2} f(n_1 b_1 T, n_2 b_2 T) y(x_1 \boxminus n_1 b_1 T, x_2 \boxminus n_2 b_2 T)$$

By (6.3) we obtain  $y(x_1 \boxminus n_1 b_1 T, x_2 \boxminus n_2 b_2 T) = \varrho_1 + \varrho_2 \varrho_3$  where  $\varrho_1, \varrho_2, \varrho_3$ , respectively, are

$$\begin{aligned} &\frac{T^2 b_1 |b_2|}{\pi^2} \cos \left( \frac{a_2 b_2 n_2^2 T^2}{2} - \frac{a_2 x_2^2}{2b_2} \right) \frac{\sin(n_1 \pi - \frac{\pi x_1^2}{b_1 T}) \sin(n_2 \pi - \frac{\pi x_2^2}{b_2 T})}{(n_1 b_1 T - x_1)(n_2 b_2 T - x_2)} \mathbf{e}^{\mathbf{i}(\frac{a_1 b_1 n_1^2 T^2}{2} - \frac{a_1 x_1^2}{2b_1})}, \\ &\text{erf} \left( \frac{|b_1 b_2|^{\frac{-1}{2}}}{2T} (2d_1 \pi - b_1 n_1 T^2 - x_1 T) \right) + \text{erf} \left( \frac{|b_1 b_2|^{\frac{-1}{2}}}{2T} (2d_1 \pi + b_1 n_1 T^2 + x_1 T) \right) \end{aligned}$$

and

$$\left( \frac{|b_1|}{\pi} \right)^{\frac{3}{2}} \frac{T^2 |d_1|^{\frac{-1}{2}}}{4} \sin \left( \frac{a_2 b_2 n_2^2 T^2}{2} - \frac{a_2 x_2^2}{2b_2} \right) \frac{\sin(n_2 \pi - \frac{\pi x_2^2}{b_2 T})}{n_2 b_2 T - x_2} \mathbf{e}^{\mathbf{i}(\frac{a_1 b_1 n_1^2 T^2}{2} + \frac{a_1 x_1^2}{2b_1} - \frac{(b_1 n_1 T + x_1)^2}{4b_1 d_1})} \mathbf{j}.$$

## 7 Conclusion

In this paper, we introduced the GSE associated with QFT. The GSE formula illustrates how a bandlimited quaternion valued signal can be recovered from the samples of system output signals. This has been realized by taking advantage of generalized translation and convolution. Moreover, we have further discussed the sampling formula for  $\sigma$ -bandlimited quaternion valued signal in quaternion linear canonical transform sense.

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